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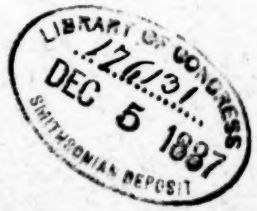

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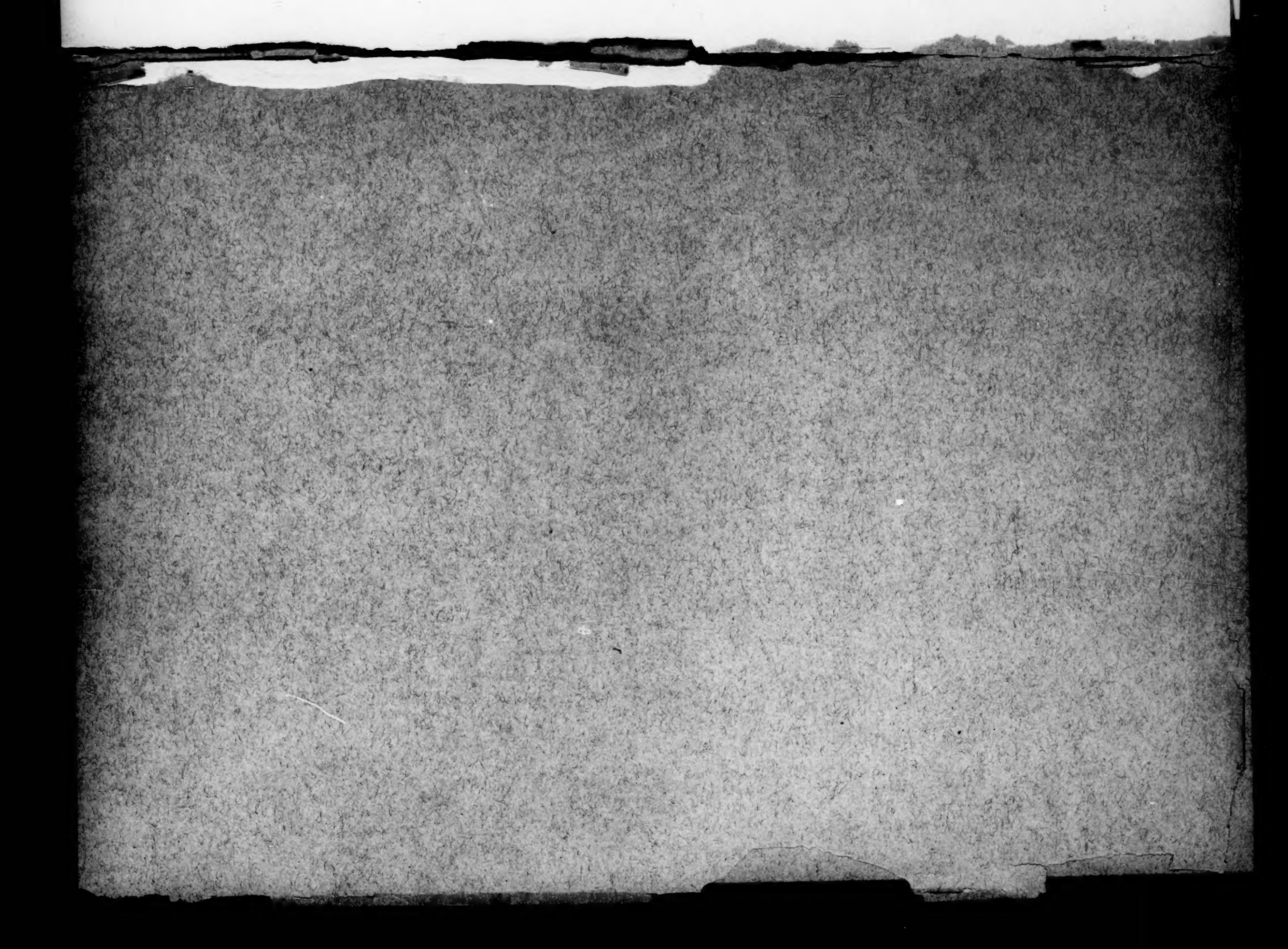

ANNALS OF MATHEMATICS.

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ON THE CONDITIONED COOLING AND THE CUBICAL CONTRACTION OF A HOMOGENEOUS SPHERE.*

By Mr. R. S. WOODWARD, Washington, D. C.

13. In the former paper (ANNALS OF MATHEMATICS, Vol. III. No. 3) we have discussed the problem of the "free cooling" of a homogeneous sphere. In the present paper it is proposed to consider what we have, for brevity, called the case of "conditioned cooling of a homogeneous sphere," or the cooling of such a sphere in a diathermous medium. In addition, we shall give an application of the theory of cooling in determining under certain limitations the consequent cubical contraction in large spheres, this problem being of special interest with reference to the secular contraction of the earth.

The physical circumstances, and the mathematical features of the solution of the problem of conditioned cooling have been very fully and ably set forth by Fourier,† Poisson,‡ and Riemann.§ So far as the purely analytical requirements are concerned, their works, especially that of Poisson, will be found exhaustive. We need, therefore, only to refer to them for a complete exposition of the elementary considerations which lead to their final result, our object being to adapt that result to the requirements of the computer, having in mind always an application of the theory to the earth. It will be desirable, however, for the sake of clearness and definiteness to make a formal statement of the fundamental assumptions and relations involved in the steps of the investigation which precede those we have in view.

* Since this paper is essentially a continuation of the one on "The Free Cooling of a Sphere," published in Vol. III. No. 3 of the ANNALS OF MATHEMATICS, the section numbers and those designating the principal equations are made continuous with the corresponding numbers of the earlier paper. The notation introduced in the latter has been adhered to as far as possible.

† Théorie Analytique de la Chaleur. Paris, 1822. Analytical Theory of Heat, by Joseph Fourier. Translated, with notes, by Alexander Freeman. Cambridge, 1878.

‡ Théorie Mathématique de la Chaleur. Paris, 1835.

§ Partielle Differentialgleichungen und deren Anwendung auf physikalische Fragen. Vorlesungen von Bernhard Riemann, für den Druck bearbeitet und herausgegeben von Karl Hattendorff. Dritte Auflage. Braunschweig, 1882.

14. The assumptions made are four, viz.: first, a constant coefficient of diffusion for the entire mass of the sphere; second, an initial uniform excess in temperature above that of the surrounding medium of the entire mass of the sphere; third, a rate of flow of heat from the surface of the sphere to the medium proportional to their difference in temperature; and fourth, a constant temperature for the surrounding medium.

As in the previous paper, let r be the radius of any spherical shell whose thickness is dr, u the excess in temperature over that of the surrounding medium of this shell at the time t after the initial epoch, and a^2 the coefficient of diffusion. Then these quantities must satisfy the partial differential equation

$$\frac{\partial (ru)}{\partial t} = a^2 \frac{\partial^2 (ru)}{\partial r^2}, \tag{29}$$

and the solution of the problem consists in determining the integral of this equation, (ru) being regarded as the unknown quantity, subject to certain relations which we proceed to set down. If u_0 be the initial uniform excess of the temperature of the sphere over that of the surrounding medium, we must have

$$ru = ru_0 \text{ for } t = 0. \tag{30}$$

In order that u may not be infinite when $r = 0$, we must have

$$ru = 0 \text{ for } r = 0. \tag{31}$$

It remains to express the characteristic condition of the problem, namely, that which defines the emission of heat at the surface of the sphere. If k be the coefficient of conductivity for the sphere, the quantity of heat which passes across a unit of its surface in an element of time is expressed by the value of $-k\frac{\partial u}{\partial r}$ when $r = r_0$, the radius of the surface of the sphere. On the other hand, this quantity of heat is equal to that which the surrounding medium can carry away, or according to our third and fourth assumptions, to Hu, H^* being a constant and u the temperature corresponding to $r = r_0$. Hence we have

$$-k\frac{\partial u}{\partial r} = Hu,$$

or, if we put $H/k = h$,

$$\frac{\partial u}{\partial r} + hu = 0 \text{ for } r = r_0.$$

But since $u = \frac{(ru)}{r}$, $\qquad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial (ru)}{\partial r} - \frac{u}{r}$,

*This quantity is called by Sir W. Thomson the emissivity. It is the quantity of heat emitted per unit of time, per unit of surface, per unit of excess of temperature; and may be due to conduction, convection, or to both. See §71, Article Heat, Encyclopedia Britannica.

which, substituted in the above relation, gives

$$\frac{\partial (ru)}{\partial r} + (rh - 1)\, u = 0 \text{ for } r = r_0. \tag{32}$$

Now a particular integral of (29) satisfying the relation (31) is

$$ru = C' e^{-\alpha^2 t \lambda^2} \sin \lambda r, \tag{33}$$

in which C' is a constant, and λ a factor such that

$$\frac{\partial (ru)}{\partial r} = C' \lambda e^{-\alpha^2 t \lambda^2} \cos \lambda r$$

and

$$u = C' r^{-1} e^{-\alpha^2 t \lambda^2} \sin \lambda r$$

will satisfy (32). Substituting these values in (32) we have λ defined by the following transcendental equation: —

$$r_0 \lambda \cos r_0 \lambda + (r_0 h - 1) \sin r_0 \lambda = 0. \tag{34}$$

This equation has an infinite number of unequal real roots,* and since any one of them will satisfy (33), the complete integral of (29) must consist of a series of terms of the form (33). That is,

$$ru = \sum_{n=1}^{n=\infty} C'_n e^{-\alpha^2 t \lambda_n^2} \sin \lambda_n r. \tag{35}$$

In this, $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the successive roots of (34). The constants $C'_1, C'_2, C'_3, \ldots$ must be such that (35) will satisfy the initial relation (30); i. e. making $t = 0$ in (35), the relation

$$ru_0 = \sum_{n=1}^{n=\infty} C'_n \sin \lambda_n r.$$

The process of determining C'_n from this equation consists in multiplying both sides by $\sin \lambda_n r dr$ and integrating between the limits 0 and r_0, account being taken of the relation (34). We shall not go through the steps of this process, referring the reader to Poisson and Riemann for the details, but shall pass immediately to one of the usual final forms for (35). This form is†

$$ru = \frac{2u_0}{r_0} \sum_{n=1}^{n=\infty} \frac{1}{\lambda_n^2} \cdot \frac{r_0^2 \lambda_n^2 + (r_0 h - 1)^2}{r_0^2 \lambda_n^2 + r_0 h (r_0 h - 1)} e^{-\alpha^2 t \lambda_n^2} (\sin r_0 \lambda_n - r_0 \lambda_n \cos r_0 \lambda_n) \sin r \lambda_n. \tag{36}$$

15. The expression (36) satisfies all of the relations (29) to (32), because each of its terms satisfies them, and is the complete solution of the problem. But, though it meets all the demands of pure analysis, its form is quite repulsive to

* See any of the works first cited above.

† See Riemann, Differentialgleichungen, §69. Other equivalent forms are given by Fourier, Poisson, and Riemann.

the computer or to any one who may desire to use it, considering that the quantities λ_n must be determined from the transcendental equation (34). Some attempts have been made to simplify this expression, but the results are not satisfactory, inasmuch as they require either a time so great that we may neglect all terms of the series after the first, or a very small radius of the sphere, or a very great radius of the sphere. No attempt, so far as we know, has been made to transform the series into another more converging or tractable, though Poisson has summed it to a limited degree of approximation, as we shall presently explain. Neither is the improvement we have to suggest entirely satisfactory, for it is restricted to the case where $r_0 h$ of (34) is greater than unity,* but this is the case with which we are most particularly concerned and that to which we now direct attention.

16. The quantity h in (34) is of -1 dimension in length,† and hence $r_0 h$ is of zero dimension or an abstract number.

For brevity, let us put

$$x = 1 / (r_0 h) \quad \text{and} \quad y_n = r_0 \lambda_n.$$

Then equation (34) gives

$$\tan y_n = -\frac{x y_n}{1 - x}. \tag{37}$$

Likewise, if we make the same substitutions in (36), it may be put in the more compact form

$$ru = 2 r_0 u_0 \sum_{n=1}^{n=\infty} \frac{1}{y_n^2} \cdot \frac{(1-x)^2 + x^2 y_n^2}{(1-x) + x^2 y_n^2} \cdot \frac{\sin y_n}{x} \cdot e^{-a^2 t \lambda_n^2} \sin \lambda_n r. \tag{38}$$

Now it will be observed that when $x = 0$, $y_n = n\pi$, $\lambda_n = n\pi / r_0$, and

$$\frac{\sin y_n}{x} = \frac{0}{0} = -n\pi \cos n\pi;$$

hence (38) becomes

$$ru = \frac{2 r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} e^{-a^2 t (n\pi / r_0)^2} \sin n\pi \frac{r}{r_0}.$$

This agrees with equation (10), which expresses the solution in the case of free cooling. The case of conditioned cooling, therefore, approximates to that of free cooling in proportion to the smallness of the number x. This number may be small by reason of the largeness of either r_0 or h, or of both. From cer-

*It is obvious, however, that a slight modification of the process followed in the text may be used in treating the case where $r_0 h < 1$.

†See Fourier's Section IX. chap. II. p. 130, Freeman's translation. Our h is the same as Fourier's h / K of the page just cited, but the same as his h defined on p. 268.

tain observations of underground temperatures in the garden of the Paris Observatory, Poisson concluded that for the earth in the vicinity named, h is a little greater than 1, using the metre as space unit.* This would make $x = 1/(r_0 h)$ about 1/6,300,000. Whether this value is nearly correct as applied to the whole surface of the earth may be doubted, but x might be 10,000 times as great without materially affecting the applicability to the earth of the process we shall develop. The correctness of that process only requires that $x < 1$.

It is proposed, now, to expand the second member of (38) by Maclaurin's theorem in a series of ascending powers of x. As a preliminary, though not essential part of this work, we may expand y_n of (37) as a function of x and arrive thus at a formula of use in a future numerical application.

17. When $x = 0$, (37) gives $y_n = n\pi$, and this is the first term in the expansionof y_n. To get the coefficients of the terms involving powers of x, we need $\frac{d^i y_n}{dx^i}$ for $x = 0$, i being any positive integer. If we differentiate (37) once, and then eliminate $\sec^2 y_n$, the result is

$$(1 - x + x^2 y_n^2)\frac{dy_n}{dx} + y_n = 0.$$

If now we put $Y = y_n^2 \frac{dy_n}{dx}$, and differentiate the above equation successively, we find that for $x = 0$

$$\frac{d^i y_n}{dx^i} = (i-2)\frac{d^{i-1} y_n}{dx^{i-1}} - (i-1)(i-2)\frac{d^{i-3} Y}{dx^{i-3}}. \qquad (39)$$

By a duplicate application of the theorem of Leibnitz, we can derive a general expression for the $(i-3)$th differential coefficient of y with respect to x, but its usefulness hardly justifies the space required by its rather complex character. We shall simply write down the first six values of (39). They are

$$\begin{aligned}
\frac{d^i y_n}{dx^i} &= -\quad n\pi \quad \text{for } i = 1,\\
&= \quad 0 \quad\quad\quad = 2,\\
&= +\quad 2(n\pi)^3 \quad = 3,\\
&= -\quad 8(n\pi)^3 \quad = 4,\\
&= -\quad 24(n\pi)^5 \quad = 5,\\
&= +\quad 384(n\pi)^5 \quad = 6.
\end{aligned}$$

Hence by Maclaurin's theorem we have

$$y_n = n\pi(1 - x + \tfrac{1}{3}n^2\pi^2 x^3 - \tfrac{1}{3}n^2\pi^2 x^4 - \ldots). \qquad (40)$$

*Théorie de la Chaleur, p. 502. The value given is $b = h = 1.05719$.

Neglecting terms involving powers of x higher than the 4th, this equation may be written

$$\begin{aligned} y_n &= n\pi\,(1-x)\,(1+\tfrac{1}{3}n^2\pi^2x^3) \\ &= n\pi - n\pi x\,\{1-\tfrac{1}{3}n^2\pi^2x^2\,(1-x)\}, \end{aligned} \tag{41}$$

one of which forms will be found convenient for logarithmic computation of the smaller values of y_n. For the larger values, the series (40) will, in general, prove too slowly converging, but it is not necessary to calculate such values. Our only use for the smaller values is a numerical verification.

18. To develop (38) we observe that each of the seven factors under the sign of summation, is a function of x. We might, therefore, expand these factors separately and then find their product. On the other hand, we might get the differential coefficients of (ru) with respect to x by direct differentiation. We shall adopt a combination of these two processes.

First, develop the product $x^{-1}\sin y_n$. By means of (40), we readily find

$$x^{-1}\sin y_n = -\,n\pi\cos n\pi\,(1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ .\ .\ .\ ).$$

Then, if we put $y_n = n\pi y$, where y is the series within the parentheses of (40), we see that

$$y_n^{-2}\,x^{-1}\sin y_n = \frac{(-1)^{n+1}}{n\pi}\,y^{-2}\,(1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ .\ .\ .\ ).$$

Now, for brevity, let

$$X_1 = (1-x)^2 + x^2y_n^2,$$
$$X_2 = 1-x\ \ +x^2y_n^2,$$

and

$$X_3 = (1-\tfrac{1}{2}n^2\pi^2x^2+\tfrac{1}{3}n^2\pi^2x^3+\ .\ .\ .\ ).$$

Then, if we denote the n^{th} term of the series under the sign of summation in (38) by T_n, and differentiate $\log T_n$, the result is

$$\frac{1}{T_n}\frac{\partial T_n}{\partial x} = -\,2y^{-1}\frac{dy}{dx} + X_1^{-1}\frac{dX_1}{dx} - X_2^{-1}\frac{dX_2}{dx} + X_3^{-1}\frac{dX_3}{dx}$$
$$-\,2a^2t\lambda_n\frac{d\lambda_n}{dx} + r\cot\lambda_n r\,\frac{d\lambda_n}{dx}.$$

But from (40), we have when $x=0$, $y=1$, and $\dfrac{dy}{dx} = -1$. Also, since $\lambda_n = \dfrac{y_n}{r_0}$,

$$\lambda_n = \frac{n\pi}{r_0} \quad\text{and}\quad \frac{d\lambda_n}{dx} = -\frac{n\pi}{r_0}.$$

Therefore, when $x=0$, we find

$$\frac{\partial T_n}{\partial x} = T_n\left[1+2t\left(\frac{an\pi}{r_0}\right)^2 - \frac{r}{r_0}\,n\pi\cot n\pi\frac{r}{r_0}\right]. \tag{42}$$

The succeeding complete values of the differential coefficients grow rapidly in

complexity. Their derivation, however, is tedious rather than difficult. It will suffice, for our purposes, to write down the special value of the second differential coefficient, viz.:

$$\frac{\partial^2 T_n}{\partial x^2} = T_n \left\{ \begin{array}{l} 4 - n^2\pi^2 \left(1 + \dfrac{r^2}{r_0^2}\right) + 2t \left(\dfrac{an\pi}{r_0}\right)^2 \\ + 4t^2 \left(\dfrac{an\pi}{r_0}\right)^4 \\ - \left[2 \dfrac{r}{r_0} n\pi + 4a^2 tr \left(\dfrac{n\pi}{r_0}\right)^3\right] \cot n\pi \dfrac{r}{r_0} \end{array} \right\}. \qquad (43)$$

Now we observe that when $x = 0$,

$$T_n = \frac{(-1)^{n+1}}{n\pi} e^{-a^2 t (n\pi / r_0)^2} \sin n\pi \frac{r}{r_0}, \qquad (a)$$

$$T_n \frac{r}{r_0} n\pi \cot n\pi \frac{r}{r_0} = -r \frac{\partial T_n}{\partial r}^*, \qquad (b)$$

$$2T_n t \left(\frac{an\pi}{r_0}\right)^2 = -2t \frac{\partial T_n}{\partial t}, \qquad (c)$$

$$T_n n^2\pi^2 \left(1 + \frac{r^2}{r_0^2}\right) = -(r_0^2 + r^2) \frac{\partial^2 T_n}{\partial r^2}, \qquad (d)$$

$$4T_n t^2 \left(\frac{an\pi}{r_0}\right)^4 = +4t^2 \frac{\partial^2 T_n}{\partial t^2}, \qquad (e)$$

$$4T_n a^2 rt \left(\frac{n\pi}{r_0}\right)^3 \cot n\pi \frac{r}{r_0} = +4rt \frac{\partial^2 T_n}{\partial r \partial t}. \qquad (f)$$

For brevity, put S for the value of the second member of (38) when $x = 0$, this value being the same as that given by (10), (19), or (20). Then, observing the effect of the sign of summation in (38), equations (42) and (43) in connection with the equivalents in (a) to (f) give

$$\frac{\partial (ru)}{\partial x} = S + r \frac{\partial S}{\partial r} - 2t \frac{\partial S}{\partial t}, \qquad (44)$$

$$\frac{\partial^2 (ru)}{\partial x^2} = \left\{ \begin{array}{l} 4S - 2r \dfrac{\partial S}{\partial r} - 2t \dfrac{\partial S}{\partial t} \\ + 4rt \dfrac{\partial^2 S}{\partial r \partial t} + (r_0^2 + r^2) \dfrac{\partial^2 S}{\partial r^2} \\ + 4t^2 \dfrac{\partial^2 S}{\partial t^2} \end{array} \right\}. \qquad (45)$$

*To get the signs of the second members of (b) and (f), one must note that $\partial T_n / \partial r$ and $d\lambda_n / dx$ have opposite signs.

Finally, by Maclaurin's series we get

$$ru = S + \frac{\partial (ru)}{\partial x} x + \tfrac{1}{2} \frac{\partial^2 (ru)}{\partial x^2} x^2 + \ldots \qquad (46)$$

19. The partial differential coefficients involved in the second members of (44) and (45) are easily calculated. If we write, for brevity,

$$Q' = \sum_{n=1}^{n=\infty} \left\{ \begin{matrix} e^{-(2m_0 n - m_0 - m)^2} \\ + e^{-(2m_0 n + m_0 + m)^2} \end{matrix} \right\}, \qquad (47)$$

$$Q'' = \sum_{n=1}^{n=\infty} \left\{ \begin{matrix} (2m_0 n - m_0 - m)\ e^{-(2m_0 n - m_0 - m)^2} \\ - (2m_0 n + m_0 + m)\ e^{-(2m_0 n + m_0 + m)^2} \end{matrix} \right\}, \qquad (48)$$

$$Q''' = \sum_{n=1}^{n=\infty} \left\{ \begin{matrix} (2m_0 n - m_0 - m)^2\, e^{-(2m_0 n - m_0 - m)^2} \\ + (2m_0 n + m_0 + m)^2\, e^{-(2m_0 n + m_0 + m)^2} \end{matrix} \right\}, \qquad (49)$$

$$Q^{\mathrm{iv}} = \sum_{n=1}^{n=\infty} \left\{ \begin{matrix} (2m_0 n - m_0 - m)^3\, e^{-(2m_0 n - m_0 - m)^2} \\ - (2m_0 n + m_0 + m)^3\, e^{-(2m_0 n + m_0 + m)^2} \end{matrix} \right\}, \qquad (50)$$

m_0 and m being defined by equations (18), we get from (23), (24), and (25)

$$r \frac{\partial S}{\partial r} = + ru_0 \left(1 - \frac{r_0}{a\sqrt{(\pi t)}} Q' \right), \qquad (51)$$

$$2t \frac{\partial S}{\partial t} = - \frac{2 r_0 u_0}{\sqrt{\pi}} Q'', \qquad (52)$$

$$(r_0^2 + r^2) \frac{\partial^2 S}{\partial r^2} = - \frac{r_0 (r_0^2 + r^2) u_0}{a^2 t \sqrt{\pi}} Q'', \qquad (53)$$

$$4t^2 \frac{\partial^2 S}{\partial t^2} = + \frac{r_0 u_0}{\sqrt{\pi}} (6Q'' - 4Q^{\mathrm{iv}}), \qquad (54)$$

$$4rt \frac{\partial^2 S}{\partial r \partial t} = + \frac{r_0 r u_0}{a\sqrt{(\pi t)}} (2Q' - 4Q'''). \qquad (55)$$

These will be applicable for all but the latest stages of cooling, and the corresponding derivatives from (10) will become applicable when these fail to be sufficiently converging.

The solution for conditioned cooling is thus made to depend on that for the ideal case of free cooling; and the expression (38), which is practically useless

(on account of its slow convergence) for small values of the time is transformed into (46), which is available for any value of the time. We have limited the expansion to terms of the second order in x, believing that this is more than sufficient for any application of the theory to the earth; but the higher terms of the series may be easily derived if needed.

20. To complete the solution, it remains to show how to determine the temperature at the centre of the sphere. When we make $r = 0$, the expression for u from (46) assumes the ambiguous form $\frac{0}{0}$; but the usual process of differentiation gives

$$u = \frac{\partial S}{\partial r} + \frac{\partial\left[\frac{\partial (ru)}{\partial x}\right]}{\partial r}\, x + \tfrac{1}{2}\, \frac{\partial\left[\frac{\partial^2 (ru)}{\partial x^2}\right]}{\partial r}\, x^2 + \ldots \qquad (56)$$

$r = 0.$

From equation (26) we have

$$\frac{\partial S}{\partial r} = u_0\left[1 - \frac{4m_0}{\pi} \sum_{n=1}^{n=\infty} e^{-m_0^2 (2n-1)^2}\right]. \qquad (57)$$

$r = 0.$

The differential of (44) gives

$$\frac{\partial\left[\frac{\partial (ru)}{\partial x}\right]}{\partial r} = 2\,\frac{\partial S}{\partial r} - 2t\,\frac{\partial^2 S}{\partial r\,\partial t}. \qquad (58)$$

$r = 0.$

From (55) we have

$$2t\,\frac{\partial^2 S}{\partial r\,\partial t} = \frac{r_0 u_0}{a_1 \sqrt{(\pi t)}}\,(Q' - 2Q'''), \qquad (59)$$

in which, of course, r must be made zero for the special value in (58).

The special value of the differential of (45) with respect to r, required by the second term of (56), may be easily written down and worked out numerically for any assumed data, but the paucity of practical applications makes this work unnecessary.*

* In note C of the supplement to his great work (see full title, p. 129) Poisson has summed the series (38) to terms of the first order in x. He does not separate his result into the term independent of and that dependent on x, but gives an expression for the sum of the two terms. He deduces several special results dependent on the assumption (a) that $x = 0$, or (b) that the strata whose temperatures are considered are near the surface of the sphere. Among these results is an expression for the temperature at the centre of the sphere under the assumption that $x = 0$. This expression agrees with our expression (26) or (57); but the casual reader would not observe the agreement; and, curiously enough, Poisson himself was unaware that the difference of two definite integrals in his expression is exactly zero. Using our notation,

21. Since no set of formulæ presenting any degree of complexity, or requiring many steps in their derivation, can be regarded as entirely trustworthy until they have been subjected to the test of actual application, we propose in the present and next sections to work out in detail a numerical example. For this purpose we shall assume such a value of the time that we can compute ru from (38) as well as from (46). The equivalence of the two forms will thus be tested, as well as the correctness of the signs of the several terms in the second members of (44) and (45).

We shall assume $m_0 = r_0 / 2a\sqrt{t} = \pi / 2$, a value used also in the illustrative example of §11. This gives

$$t = \left(\frac{r_0}{a\pi} \right)^2,$$

which, for the earth, if we use Sir W. Thomson's value of a^2, is, as stated in §3, about 100,000,000,000 years, With this value of the time, the first three terms of (38) will suffice to give ru in terms of $r_0 u_0$ to seven places of decimals. We shall also assume $x = 10^{-3}$ and $r = \frac{1}{2} r_0$ and compute first by means of (38).

22. To begin with, we need the first three roots of the transcendental equation (37). These we get from the first of (41), which gives with sufficient accuracy $\log y_n = \log (n\pi) + \log (1 - x) + \frac{1}{3}\mu (n\pi)^2 x^3$; μ being the modulus of the common system of logarithms. This gives

$$\log y_1 = 0.4967154,$$
$$\log y_2 = 0.797745,$$
$$\log y_3 = 0.973837.$$

For use in $\sin y_n$ and $\sin \lambda_n r$, it is best to compute the supplements $(n\pi - y_n)$.

he has for the fall of temperature at the centre of the sphere (see p. 52 of the supplement)

$$u_0 - u = \frac{2u_0}{\sqrt{\pi}} \left[\int_{m_0}^{\infty} e^{-z^2} dz + 2m_0 \sum_{n=1}^{n=\infty} e^{-m_0^2 (2n-1)^2} - \sum_{n=1}^{n=\infty} \int_{-m_0}^{+m_0} e^{-(2m_0 n + z)^2} dz \right],$$

while our equation (26) gives only the middle term of this expression. But if in the last definite integral we replace $2m_0 n + z$ by z, the result is

$$\sum_{n=1}^{n=\infty} \int_{m_0 (2n-1)}^{m_0 (2n+1)} e^{-z^2} dz = \int_{m_0}^{3m_0} e^{-z^2} dz + \int_{3m_0}^{5m_0} e^{-z^2} dz + \ldots$$
$$= \int_{m_0}^{\infty} e^{-z^2} dz,$$

which is identical with the first definite integral. Hence the above expression is the same as (26). Poisson, after reaching his equation, observes that

$$\int_{-m_0}^{+m_0} e^{-(2m_0 n + z)^2} dz < 2m_0 e^{-m_0^2 (2n-1)^2},$$

and proceeds to draw some conclusions which are qualitatively correct, but not susceptible of definite numerical evaluation.

These may be accurately computed from the second of (41), which gives in seconds of arc

$$\pi - y_1 = 647''.97787,$$
$$2\pi - y_2 = 1295.983,$$
$$3\pi - y_3 = 1943.94.$$

These and the above values of $\log y_n$ will be found to satisfy (37).

We can now compute all the factors in (38). The logarithms of the second algebraic factor and those of the factor $x^{-1} \sin y_n$ are found to be

$\log X_1 X_2^{-1}$	$\log x^{-1} \sin y_n$	
9.9995655 — 10,	0.4971479,	for $n = 1$,
9.999565 — 10,	0.798170,	$= 2$,
9.999566 — 10,	0.974252,	$= 3$.

Since $\lambda_n = y_n / r_0$, we have with our assumed value of t

$$e^{-a^2 t \lambda_n^2} = e^{-(y_n \cdot \pi)^2},$$

and hence

$\log e^{-a^2 t \lambda_n^2} = 9.5665736 - 10$, for $n = 1$,
$= 8.266294 - 10$, $= 2$,
$= 6.099162 - 10$, $= 3$.

Likewise, for the last factor we have $\sin \lambda_n r = \sin y_n r / r_0 = \sin \frac{1}{2} y_n$. Hence

$\log \sin \frac{1}{2} y_n = 9.9999995 - 10$, for $n = 1$,
$= 7.798170 - 10$, $= 2$,
$= 9.999995_n - 10$, $= 3$.

Summing the logarithms of the factors of the several terms we get

	log		No.
1st term,	9.0698557 — 10,	+	0.11745073,
2d	4.965681 — 10,	+	. 0924,
3d	5.125300_n — 10,	—	. 1334.

The sum of the terms is 0.11744663. Multiplying this by 4 we get as the result from (38),

$$u = 0.4697865 u_0.$$

23. The details of the computation by equation (46) are as follows: —

The value of twice the first term, S, has already been computed in the example of §11; i. e.

$$S = 0.2341732 r_0 u_0,$$

using the larger of the two values there given, since a more extended computation shows that to be the more precise value.

We next need the values of the Q's which enter the second members of (51) to (55). Since by (18), $m = r/2a_1\sqrt{t}$, and since for this example we have assumed $m_0 = r_0/2a_1\sqrt{t} = \pi/2$ and $r = \frac{1}{2}r_0$, we have for use in (47) to (50)

$$\begin{aligned} m_0 - m &= \tfrac{1}{4}\pi, \quad & m_0 + m &= \tfrac{3}{4}\pi, \\ 3m_0 - m &= \tfrac{5}{4}\pi, \quad & 3m_0 + m &= \tfrac{7}{4}\pi, \\ 5m_0 - m &= \tfrac{9}{4}\pi, \quad & 5m_0 + m &= \tfrac{11}{4}\pi, \\ &\cdots\cdots\cdots & &\cdots\cdots\cdots \end{aligned}$$

Hence we have, to terms of an order which may be neglected, the following system of equations in which the exponential quantities are factors respectively of the several coefficients in the vertical rows:

$$\begin{aligned} Q' &= 1 \quad e^{-(\frac{1}{4}\pi)^2} + \quad 1 \quad e^{-(\frac{3}{4}\pi)^2} + \quad 1 \quad e^{-(\frac{5}{4}\pi)^2} + \cdots \\ Q'' &= \tfrac{1}{4}\pi \qquad - \tfrac{3}{4}\pi \qquad + \tfrac{5}{4}\pi \qquad - \cdots \\ Q''' &= (\tfrac{1}{4}\pi)^2 \qquad + (\tfrac{3}{4}\pi)^2 \qquad + (\tfrac{5}{4}\pi)^2 \qquad + \cdots \\ Q^{\mathrm{iv}} &= (\tfrac{1}{4}\pi)^3 \qquad - (\tfrac{3}{4}\pi)^3 \qquad + (\tfrac{5}{4}\pi)^3 \qquad - \cdots \end{aligned}$$

These give, by an easy computation,

$$\begin{aligned} Q' &= +\,0.54352, \\ Q'' &= +\,0.41469, \\ Q''' &= +\,0.35443, \\ Q^{\mathrm{iv}} &= +\,0.21069. \end{aligned}$$

Introducing these factors and the values of t and r in equations (51) to (55), we find the following equivalents, in which $r_0 u_0$ is a common factor of the second members:—*

$$r\frac{\partial S}{\partial r} = +\,0.01832\, r_0 u_0,$$

$$2t\frac{\partial S}{\partial t} = -\,0.46793,$$

$$(r_0^2 + r^2)\frac{\partial^2 S}{\partial r^2} = -\,2.886,$$

$$4t^2\frac{\partial^2 S}{\partial t^2} = +\,0.928,$$

$$4rt\frac{\partial^2 S}{\partial r\,\partial t} = -\,0.293.$$

* The numerical values of all these differential coefficients have been checked by means of the equivalent forms obtained from equation (10).

By means of the last values and the above value of S we get from (44)

$$\frac{\partial (ru)}{\partial x} = + 0.72042\, r_0 u_0.$$

Likewise from (45)

$$\frac{\partial^2 (ru)}{\partial x^2} = - 0.883\, r_0 u_0.$$

With these values and with $x = 10^{-3}$, we have from (46)

$$u = 2u_0 (0.2341732 + 0.00072042 - 0.00000044)$$
$$= 0.4697864 u_0.$$

This agrees with the value obtained from (38) as closely as 7-place tables of logarithms will permit, the discrepancy being one unit in the last place.

It is scarcely necessary to remark in this connection that the precision of the preceding computations is of little importance except as a test of the correctness of the formulæ derived and as an illustration of the process which might be followed if our data for practical applications were sufficiently trustworthy. For all applications to the earth, computation of results by the preceding formulæ may be properly limited, in general, to three or four places of decimals, since defects in the best data at present attainable and probable inaccuracies in our fundamental assumptions (of §14), render those results uncertain even in the first place of decimals. For these reasons, also, and in view of the probability that the quantity x is less than 1 / 1000, it is believed that the solution for free cooling is an ample approximation in all applications to the earth.

24. In this and the following sections it is proposed to determine the amount of cubical contraction of a homogeneous sphere on the assumption of free cooling, and on the additional assumption that the coefficient of contraction is constant. These assumptions will give correctly the first approximation (or the term independent of x) in the case of a homogeneous sphere of moderately high initial temperature, and probably as good an approximation as is at present attainable for the secular contraction of the earth.*

If ε denote the coefficient of cubical contraction and ΔV the total change in volume during the time t after the initial epoch, we shall have

$$\Delta V = 4\pi\varepsilon \int_0^{r_0} (u_0 - u)\, r^2 dr, \qquad (60)$$

in which the symbols under the sign of integration have the same meaning as

*The consideration of the mechanical effects of secular contraction forms no part of our present work, though one of the principal objects of the analysis of this and the preceding paper is to furnish a basis for the investigation of those effects.

heretofore. The fall in temperature, $u_0 - u$, is given by equation (21) or (22), but for our present purposes it will be convenient to use another expression easily derived from (10). If in the latter, we make t first zero, and then t, and take the difference between the two expressions the result is

$$r(u_0 - u) = \frac{2r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} (1 - e^{-a^2 t (n\pi / r_0)^2}) \sin n\pi \frac{r}{r_0}. \qquad (61)$$

Introducing this value of the fall in temperature of the shell considered, in (60) we get

$$\Delta V = 8 r_0 u_0 \varepsilon \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} (1 - e^{-a^2 t (n\pi / r_0)^2}) \int_0^{r_0} r dr \sin n\pi \frac{r}{r_0}.$$

But the integral in this equation is equal to

$$-\frac{r_0^2}{\pi} \frac{\cos n\pi}{n} = \frac{r_0^2}{\pi} \frac{(-1)^{n+1}}{n}.$$

Therefore we have as the exact solution, under the assumed conditions,

$$\Delta V = \frac{8 r_0^3 u_0 \varepsilon}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n^2} (1 - e^{-a^2 t (n\pi / r_0)^2}). \qquad (62)$$

When $t = 0$, this expression gives $\Delta V = 0$, as it should. When $t = \infty$, the quantity under the sign of summation becomes

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots = \pi^2/6;$$

and hence

$$\Delta V = \tfrac{4}{3}\pi r^3 u_0 \varepsilon,$$

which is also obviously correct.

25. For large values of the time the series in (62) is evidently rapidly converging, but for small values the convergence is too slow for practical applications. It is desirable, therefore, to transform (62) and derive an expression from which we can compute the contraction when t is small, or during the earlier stages of cooling. To this end we proceed.

Referring again to the Gamma function of §4, and making the same substitutions for α and β as in §5, the series in (62) becomes

$$\tfrac{1}{6}\pi^2 - \frac{r_0}{a\sqrt{(\pi t)}} \int_0^{\infty} e^{-r_0^2 x^2 / 4a^2 t} dx (\cos \pi x + \tfrac{1}{4} \cos 2\pi x + \tfrac{1}{9} \cos 3\pi x + \ldots),$$

in which x is merely a subject of integration and is, of course, not the x of

§§16 to 23. Now, for brevity, denote the trigonometric series in the last expression by σ_2, and let

$$\frac{d\sigma_2}{dx} = \sigma_1, \qquad \frac{d^2\sigma_2}{dx^2} = \frac{d\sigma_1}{dx} = \sigma.$$

From the series in §6 we see that

$$\sigma = \tfrac{1}{2}\pi^2 - \frac{\tfrac{1}{2}(1 - g^2)\pi^2}{1 - 2g\cos\pi x + g^2} \text{ when } g = 1.$$

Multiplying by dx, integrating between the limits 0 and x, and making $g = 1$, we get

$$\sigma_1 = \tfrac{1}{2}\pi^2(x - 1) \text{ for } 0 < x < 2.$$

Multiplying the last equation by dx and integrating again between 0 and x, there results

$$\sigma_2 = \pi^2(\tfrac{1}{6} + \tfrac{1}{4}x^2 - \tfrac{1}{2}x).$$

This gives the value of σ_2 for all values of x from 0 to 2 both inclusive, or through the first period. For succeeding periods we have

$$\begin{aligned}\sigma_2 &= \pi^2[\tfrac{1}{6} + \tfrac{1}{4}(x-2)^2 - \tfrac{1}{2}(x-2)], \text{ for } x \text{ between 2 and 4,}\\ &= \pi^2[\tfrac{1}{6} + \tfrac{1}{4}(x-4)^2 - \tfrac{1}{2}(x-4)], \text{ for } x \text{ between 4 and 6,}\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Substituting these values of σ_2 in the above integral, and writing, for brevity, $m_0 = r_0/2a\sqrt{t}$, as in equation (18), we get

$$\tfrac{1}{6}\pi^2 - 2m_0\pi^{\frac{3}{2}}\left\{\begin{aligned}&\int_0^2 e^{-m_0^2x^2}\,dx\,(\tfrac{1}{6} + \tfrac{1}{4}x^2 - \tfrac{1}{2}x)\\ &+\int_2^4 e^{-m_0^2x^2}\,dx\,[\tfrac{1}{6} + \tfrac{1}{4}(x-2)^2 - \tfrac{1}{2}(x-2)]\\ &+\int_4^6 e^{-m_0^2x^2}\,dx\,[\tfrac{1}{6} + \tfrac{1}{4}(x-4)^2 - \tfrac{1}{2}(x-4)]\\ &+\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}.$$

Collecting and arranging the terms of this expression and performing the integrations so far as possible, it reduces to

$$\frac{\pi^2}{2m_0}\left(\frac{1}{\sqrt{\pi}} - \frac{1}{4m_0}\right) + \pi^{\frac{3}{2}}\sum_{n=1}^{n=\infty}\left[m_0^{-1}e^{-(2m_0n)^2} - 2m_0n(n+1)\int_{2n}^{2n+2} e^{-m_0^2x^2}\,dx\right].$$

Substituting this value for the quantity under the sign of summation in (62) and

replacing x by z/m_0, we get finally

$$\Delta V = 8\pi r_0^3 u_0 \varepsilon \left\{ \begin{array}{l} \dfrac{1}{2m_0\sqrt{\pi}} - \dfrac{1}{8m_0^2} \\ + \sum_{n=1}^{n=\infty} \left[\dfrac{1}{m_0\sqrt{\pi}} e^{-(2m_0 n)^2} - \dfrac{2}{\sqrt{\pi}} n(n+1) \int_{2m_0 n}^{m_0(2n+2)} e^{-z^2} dz \right] \end{array} \right\}, \quad (63)$$

$$m_0 = r_0 / 2a\sqrt{t}.$$

26. The series in the right-hand member of (63) converge with extreme suddenness for small values of the time. For the case of the earth they converge so rapidly that their first terms alone will suffice for the most accurate computations (one part in a million, say) when the time is as great as $r_0^2/4a^2$, which, with Sir W. Thomson's value of a^2, is about 273,000,000,000 years. Using this value of the time we have computed the amount of the contraction by both (62) and (63). The first three terms of (62) will suffice in this case for the precision attainable with 7-place logarithms. The second term in the first series of (63) or $e^{-16}/\sqrt{\pi}$ amounts to but six units in the eighth place of decimals; and the second term of the second series, or

$$\frac{12}{\sqrt{\pi}} \int_4^6 e^{-z^2} dz,$$

amounts to but twenty-eight units in the eighth place. Omitting other details of the computation, the results are

$$\Delta V = 0.1580729 \cdot 8\pi r_0^3 u_0 \varepsilon \text{ from (62)},$$
$$= 0.1580728 \cdot 8\pi r_0^3 u_0 \varepsilon \qquad (63).$$

These results serve to verify the numerical equivalence of (62) and (63), and show that they, like (46), or (10) and the group (19) to (22), bridge completely the time interval of cooling and enable the computer to give a definite answer to any question falling within the scope of the assumed conditions.

It is worthy of remark that the contraction just computed is about

$$\Delta V = 0.95 \cdot \tfrac{4}{3}\pi r_0^3 u_0 \varepsilon;$$

that is, in the time $t = r_0^2/4a^2$, a sphere will, under the assumed circumstances, attain ninety five per cent. of its possible contraction. The temperature at the centre of the sphere at the end of this time is (see §12) about one-sixth its initial value.

In all practical applications to the earth, for any but the latest stages of cooling, we may neglect the series in (63), which becomes then simply

$$\Delta V = 8\pi r_0^3 u_0 \varepsilon \left(\frac{a}{r_0} \sqrt{\frac{t}{\pi}} - \frac{a^2 t}{2r_0^2} \right). \qquad (64)$$

From this it appears that in the earlier stages of cooling the contraction is proportional to the square root of the time.

ON DIFFERENTIAL EQUATIONS WITH PERIODIC INTEGRALS.

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The independent variable being conceived as time, a system of differential equations may be said to admit periodic integrals when the values of the dependent variables either exactly, or with approximate tendency, after a certain lapse of time, repeat their series of values. In the latter case the larger the lapse is made the more nearly is the repetition brought about. Strange as it may seem, this subject, except in the case of simply periodic integrals, is, at present, not completely understood. The text-books on differential equations are almost wholly engaged with the cases in which, by certain artifices, the integration can be accomplished in finite terms or reduced to quadratures. In the treatment of physical problems, however, equations of this sort are rarely met with. Far more frequently it is found that methods of approximation must be resorted to. Cauchy appears to be the author who has done most for the elucidation of this part of the subject. His memoirs are in his later *Exercises* and in the volumes of the *Comptes Rendus* for 1856 and 1857. In this article I propose to show how simply periodic integrals arise and afterwards to illustrate the general theory by treating a problem relating to the motion of a system of points.

I.

Having the independent variable t, and the two dependent variables x and x_1, let us suppose the latter satisfy the equations

$$\frac{dx}{dt} = x_1, \quad \frac{dx_1}{dt} = \mathrm{f}(x).$$

A cross multiplication between the members of these equations gives

$$x_1 \frac{dx_1}{dt} = \mathrm{f}(x) \frac{dx}{dt}.$$

The integral of this is, C being the arbitrary constant,

$$x_1^2 = 2\int \mathrm{f}(x)\, dx + C.$$

The values of x and x_1 being known for a given value of t, we readily find the value of C proper to the special case we treat. By substituting the value of x_1 derived from this equation in the first of the differential equations we get

$$\frac{dx}{dt} = \sqrt{\left[2\int \mathrm{f}(x)\, dx + C\right]}.$$

The expression under the radical sign is a function of x; calling it X, let us consider the equation $X = 0$. Since real values of x are supposed to correspond to all values of t, X can never be negative; and, from the way the constant C was determined, it is plain that, for the given value of t, X is positive. Then in $X = 0$, let x be supposed to increase until a value $x = b$ is reached for which $X = 0$, that is to say a real root of this equation. Similarly let x diminish from the same point until a value $x = c$ is reached for which again $X = 0$, that is a second real root Then, X being positive for all values of x which lie between c and b, if the latter are non-multiple roots, X is negative for values of x which lie just outside these limits. Thus x must necessarily remain within the limits c and b. Also, in its motion, it always attains them; for suppose x is augmenting, then the radical, which forms the value of dx/dt, must be taken positively, and, from the law of continuity, must continue to be so taken until it becomes zero, that is until x arrives at the value b. But dx/dt cannot be positive beyond this point, for x cannot surpass b. Hence, after this, the radical must receive the negative sign, and, consequently, x begins to diminish. Again, from the law of continuity, this diminution is kept up until x has arrived at the value c. At this point the diminution must change into an augmentation, for x cannot fall below c. Thus the movement of x is a continuous swinging back and forth between the limits c and b.

We can put $$X = \frac{(b-x)(x-c)}{R^2},$$

R being a function of x which remains constantly positive and finite for all values of x between c and b. We can then write

$$\frac{dt}{dx} = \frac{R}{\sqrt{[(b-x)(x-c)]}}.$$

A new variable u can now be advantageously introduced in place of x. Let

$$x = a(1 - e\cos u),$$

where $a = \frac{1}{2}(b+c)$, and $e = (b-c)/(b+c)$; and u is equivalent to an integral number of circumferences when $x = c$, and augments by half a circumference when x, next following, attains the value b. Thus u, like t, augments continuously. We have

$$b - x = ae(1 + \cos u), \quad x - c = ae(1 - \cos u),$$

$$\sqrt{[(b-x)(x-c)]} = ae\sin u,$$

$$dx = ae\sin u\, du.$$

Therefore $$dt = R\,du.$$

As R is a one-valued function of x or of $a(1 - e\cos u)$, it can be expanded in the following periodic series

$$R = \frac{1}{n}[1 + a_1\cos u + 2a_2\cos 2u + 3a_3\cos 3u + \ldots],$$

n, a_1, a_2, etc. being constants, the first having the value

$$\frac{1}{n} = \frac{1}{\pi}\int_0^{\pi} R\,du.$$

Then c being an arbitrary constant,

$$n(t + c) = u + a_1\sin u + a_2\sin 2u + a_3\sin 3u + \ldots.$$

This series serves for determining t when x or u is given; but, more frequently it is x or u which is required in terms of t. It is necessary, then, to invert the series. The coefficients of the inverted series are most readily found by means of definite integrals. Let us suppose that it is required to find the periodic series, in terms of t, for a function of x and x_1 which we will denote by U. This function we assume to be always finite and continuous. The base of hyperbolic logarithms being ε, let us put

$$\zeta = n(t + c), \quad z = \varepsilon^{\zeta\sqrt{-1}}, \quad s = \varepsilon^{u\sqrt{-1}},$$

and, for brevity,

$$2S = a_1(s - s^{-1}) + a_2(s^2 - s^{-2}) + a_3(s^3 - s^{-3}) + \ldots.$$

The equation connecting z and s is

$$z = s\varepsilon^{S}.$$

We can suppose that
$$U = \sum_{i=-\infty}^{i=+\infty} A_i z^i.$$

Then
$$A_i = \frac{1}{2\pi}\int_0^{2\pi} Uz^{-i}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} Us^{-i}\varepsilon^{-iS}\,nR\,du.$$

U being $= F(x, x_1)$, we have

$$U = F\left\{a(1 - e\cos u),\quad \frac{ae\sin u}{R}\right\}$$

$$= F\left\{a[1 - \tfrac{1}{2}e(s + s^{-1})],\quad \frac{ae(s - s^{-1})}{2R\sqrt{(-1)}}\right\}.$$

Supposing that U is reduced to x, it is plain that the coefficient of z^i, in the

development of x in powers of z, is the same as the coefficient of s^i in the development of

$$a\left[1 - \tfrac{1}{2}e(s + s^{-1})\right]\left[1 + s\frac{\partial S}{\partial s}\right]\varepsilon^{-iS}$$

in powers of s.

By adopting the Besselian functions $J_\lambda^{(i)}$, we have

$$\varepsilon^{-\frac{1}{2}ia_1(s-s^{-1})} = \sum_{i=-\infty}^{i=+\infty} J_{-\frac{1}{2}ia_1}^{(i)} s^i, \quad \varepsilon^{-\frac{1}{2}ia_2(s^2-s^{-2})} = \sum_{j=-\infty}^{j=+\infty} J_{-\frac{1}{2}ia_2}^{(j)} s^{2j}, \text{ etc.;}$$

and the expression, given above, can be written

$$a\left\{\begin{array}{l} 1 - \tfrac{1}{2}a_1 e \\ \quad + \left(\dfrac{a_1 - e}{2} - \dfrac{2}{2.2}a_1 e\right)(s + s^{-1}) \\ \quad + \left(\dfrac{2}{2}a_2 - \dfrac{a_1 e}{2.2} - \dfrac{3}{2.2}a_3 e\right)(s^2 + s^{-2}) \\ \quad + \left(\dfrac{3}{2}a_3 - \dfrac{2}{2.2}a_2 e - \dfrac{4}{2.2}a_4 e\right)(s^3 + s^{-3}) \\ \quad + \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$
$$\times (\Sigma_j . J_{\lambda_1}^{(j)} s^j) . (\Sigma_j . J_{\lambda_2}^{(j)} s^{2j}) . (\Sigma_j . J_{\lambda_3}^{(j)} s^{3j}) \ldots,$$

where $\lambda_j = -\frac{1}{2}ia_j$.

However, unless the coefficients $a_1, a_2, a_3, \ldots$ decrease rapidly, this will not be a practical method of developing x in a periodic series. Generally it will be shorter to employ mechanical quadratures in obtaining the value of the definite integral. Let us suppose that

$$x = \tfrac{1}{2}\beta_0 + \beta_1 \cos\zeta + \beta_2 \cos 2\zeta + \beta_3 \cos 3\zeta + \ldots.$$

Then

$$\beta_i = \frac{2}{\pi}\int_0^\pi x \cos i\zeta \, d\zeta$$

$$= \frac{2}{\pi}\int_0^\pi a(1 - e\cos u)\left[1 + a_1\cos u + 2a_2\cos 2u + 3a_3\cos 3u + \ldots\right]\cos i\zeta \, du,$$

where, to obtain the value of ζ corresponding to a given value of u, we employ the equation

$$\zeta = u + a_1 \sin u + a_2 \sin 2u + \ldots.$$

It will be seen that this method is applicable to a much wider range of questions than the motion of planets in elliptic orbits. And the superiority of the

method of definite integrals over Lagrange's Theorem for the inversion of the series is quite manifest.

II.

In order to illustrate the preceding general theory, let us treat the problem of n material points moving about a centre under the action of central forces admitting a potential which is a function of the sum of the squares of the *radii vectores*. Each point will then move in a fixed plane and its *radius vector* will describe equal areas in equal times. Thus all will be virtually known in reference to these motions, provided we are able to express the *radii vectores* as functions of the time.

Let the *radii* be denoted as $r_1, r_2, \ldots r_n$, and the orbit longitudes, measured each from any point in its plane, as $\lambda_1, \lambda_2, \ldots \lambda_n$. For brevity, put

$$\rho^2 = r_1^2 + r_2^2 + \ldots + r_n^2.$$

Then, if the potential is represented by $\mathrm{f}(\rho)$, we shall have the two equations, representing generally all the equations of the problem,

$$\frac{d^2 r_i}{dt^2} - r_i \frac{d\lambda_i^2}{dt^2} = \mathrm{f}'(\rho)\,\frac{r_i}{\rho},$$

$$\frac{d\lambda_i}{dt} = \frac{h_i}{r_i^2},$$

h_i being the constant of areolar velocity. Consequently if we put

$$\Omega = \mathrm{f}(\rho) - \tfrac{1}{2}\Sigma \frac{h_i^2}{r_i^2},$$

the general form of the differential equations determining the *radii vectores* will be

$$\frac{d^2 r_i}{dt^2} = \frac{\partial \Omega}{\partial r_i}.$$

They have the integral, corresponding to that of living forces,

$$\Sigma \frac{dr_i^2}{dt^2} = 2(\Omega + C),$$

C being an arbitrary constant. Also we may derive

$$\Sigma r_i \frac{d^2 r_i}{dt^2} = \Sigma\, r_i \frac{\partial \Omega}{\partial r_i}.$$

By adding the last two equations,

$$\frac{d}{dt}\left(\rho \frac{d\rho}{dt}\right) = 2\mathrm{f}(\rho) + \rho\,\mathrm{f}'(\rho) + 2C,$$

an equation involving only the dependent variable ρ. Multiplying it by the factor $2\rho\frac{d\rho}{dt}$, and integrating, we get, A being an arbitrary constant,

$$\rho^2\frac{d\rho^2}{dt^2} = 2\rho^2[\mathrm{f}(\rho) + C] - A^2.$$

Whence $$t + c = \int \frac{\rho d\rho}{\sqrt{\{2\rho^2[\mathrm{f}(\rho) + C] - A^2\}}}.$$

Inverting this we shall have ρ as a function of t.

By dividing the penultimate equation by ρ^2 and differentiating, we get

$$\frac{d^2\rho}{dt^2} = \frac{\mathrm{f}'(\rho)}{\rho}\rho + \frac{A^2}{\rho^3}.$$

The general equation determining r_i is

$$\frac{d^2r_i}{dt^2} = \frac{\mathrm{f}'(\rho)}{\rho}r_i + \frac{h_i^2}{r_i^3}.$$

As ρ is now a known function of t, r_i is the only unknown in it, and, consequently, the equation by itself suffices for determining it. To put the equation in a form suitable for integration, let us eliminate $\mathrm{f}'(\rho)$ between the last two equations. We get

$$\frac{d(\rho dr_i - r_i d\rho)}{dt^2} = \frac{h_i^2}{r_i^3}\rho - \frac{A^2}{\rho^2}r_i,$$

or $$\rho^2\frac{d}{dt}\left[\rho^2\frac{d}{dt}\left(\frac{r_i}{\rho}\right)\right] = \left[\frac{h_i^2\rho^4}{r_i^4} - A^2\right]\frac{r_i}{\rho}.$$

To simplify this, we will adopt an auxiliary variable ψ, such that

$$d\psi = \frac{A}{\rho^2}dt = \frac{Ad\rho}{\rho\sqrt{\{2\rho^2[\mathrm{f}(\rho) + C] - A^2\}}}.$$

Then $$\frac{d^2\left(\frac{r_i}{\rho}\right)}{d\psi^2} = \left[\frac{h_i^2}{A^2}\frac{\rho^4}{r_i^4} - 1\right]\frac{r_i}{\rho}.$$

Whence, by integration, we derive

$$\left\{\frac{d\left(\frac{r_i}{\rho}\right)}{d\psi}\right\}^2 = 2a_i - \frac{r_i^2}{\rho^2} - \frac{h_i^2}{A^2}\frac{\rho^2}{r_i^2},$$

a_i being the arbitrary constant. By putting

$$\frac{r_i}{\rho} = \chi(u_i),$$

we get $$d\varphi = \frac{du_i}{2\sqrt{\left[2a_i u_i - u_i^2 - \frac{h_i^2}{A^2}\right]}}.$$

For convenience, adopting a new constant e_i, in place of h_i, such that $h_i^2/A^2 = a_i^2(1 - e_i^2)$ the quantity under the radical sign becomes

$$[a_i(1 + e_i) - u_i][u_i - a_i(1 - e_i)].$$

Thus, putting $u_i = a_i(1 - e_i \cos \varepsilon_i)$, ε_i being a new variable, we get $d\varphi = \frac{1}{2}d\varepsilon_i$, and thus $\varepsilon_i = 2\varphi + \alpha_i$, α_i being a constant. Thus we have, in fine,

$$\frac{r_i}{\rho} = 1/\{a_i[1 - e_i \cos(2\varphi + 2\alpha_i)]\}.$$

As we have $$\Sigma \frac{r_i^2}{\rho^2} = 1,$$

the constants a_i, e_i, and α_i satisfy the relations

$$\Sigma a_i = 1, \quad \Sigma a_i e_i \cos 2\alpha_i = 0, \quad \Sigma a_i e_i \sin 2\alpha_i = 0.$$

We thus have $2n$ independent arbitrary constants introduced by integration; the number there should be.

In order to find an expression for the longitudes, we take the general equation

$$d\lambda_i = \frac{h_i dt}{a_i \rho^2 [1 - e_i \cos 2(\varphi + \alpha_i)]}$$
$$= \frac{1/(1 - e_i^2)\, d\varphi}{1 - e_i \cos 2(\varphi + \alpha_i)}.$$

The integral of which gives

$$\tan(\lambda_i + \beta_i) = \sqrt{\left(\frac{1 + e_i}{1 - e_i}\right)} \cdot \tan(\varphi + \alpha_i),$$

β_i being the arbitrary constant.

To simplify the equations which give $t + c$ and φ, we suppose that $a(1 + e)$ is the maximum value of ρ, and $a(1 - e)$ its minimum value. Then we can adopt a variable ε such that

$$\rho = a(1 - e \cos \varepsilon).$$

Thus $d\rho = ae \sin \varepsilon\, d\varepsilon$, and we may put

$$2\rho^2[f(\rho) + C] - A^2 = K^2 a^2 e^2 \sin^2 \varepsilon,$$

where R remains constantly positive throughout the motion of ρ. Then

$$t + c = \int \frac{\rho}{R} d\varepsilon,$$

$$\varphi = \int \frac{A}{\rho R} d\varepsilon.$$

R, being a function of ρ, is also one of $a(1 - e\cos\varepsilon)$, and thus is capable of being expanded in a converging series of terms, each consisting of a constant multiplied by the cosine of a multiple of ε. Also ρ/R and $A/\rho R$ can be expanded in similar series. Then the period T, in which ρ goes through the round of its values, is given by the definite integral

$$T = \int_0^{2\pi} \frac{a(1 - e\cos\varepsilon)}{R}\, d\varepsilon,$$

and the augmentation of the variable ψ, in the same time, will be equivalent to the definite integral

$$\int_0^{2\pi} \frac{A d\varepsilon}{a(1 - e\cos\varepsilon) R}.$$

If the value of the latter is 2π, ψ will augment by a circumference while ρ goes through its period. This is the case when $f(\rho) = \mu/\rho$; but, in general, this condition is not fulfilled.

Provided that A^2 is a positive quantity, it is plain that, after ψ has gone through its period, the longitudes and latitudes, whether as seen from the centre or from any of the points, all return to the same values. The same thing is true of the ratios of the *radii vectores*. Thus the movement of the system may be conceived as taking place under the operation of two distinct causes. The first producing a revolution of all the points about the centre in closed curves and in the same time, while the second, having a different period, changes the scale of representation of the system in space.

In the preceding treatment we have supposed that A^2 is a positive quantity. When this is not the case, some modifications must be made. Let us suppose first that $A = 0$. Then we have

$$t + c = \int \frac{d\rho}{\sqrt{\{2[f(\rho) + C]\}}},$$

and we may assume

$$\psi = \int \frac{d\rho}{\rho^2 \sqrt{\{2[f(\rho) + C]\}}}.$$

Then

$$\left\{ \frac{d\left(\frac{r_i}{\rho}\right)}{d\psi} \right\}^2 = a_i - h_i^2 \frac{\rho^2}{r_i^2},$$

$$\psi + \alpha_i = \sqrt{\left(a_i \frac{r_i^2}{\rho^2} - h_i^2\right)},$$

$$\frac{r_i}{\rho} = \sqrt{\left(\frac{(\psi + \alpha_i)^2 + h_i^2}{a_i}\right)}.$$

Also
$$d\lambda_i = \frac{h_i\,dt}{r_i^2} = h_i \frac{t^2}{r_i^2}\,d\psi$$
$$= \frac{h_i a_i\,d\psi}{(\psi + \alpha_i)^2 + h_i^2},$$
$$\tan(\lambda_i + \beta_i) = \frac{a_i}{h_i}(\psi + \alpha_i).$$

In the second place, let A^2 be negative. Here it is only necessary in some places to accomplish the integrations by the aid of hyperbolic cosines instead of circular.

The differential equations of this problem, in the case where the *radii* are supposed to describe no areas, were first integrated by Binet.* But the addition, to the forces, of the terms arising from centrifugal action, much enhances the interest of the problem.

*See Liouville, *Journal de Mathématiques*, First Ser. Tome II. p. 457.

ON THE FOCAL CHORD OF A PARABOLA.

By Prof. R. H. Graves, Chapel Hill, N. C.

Let $y^2 = 4ax$ be the equation to a parabola, s its focus, and PSP' a focal chord. Let the tangent and normal at P' meet the diameter through P at M and N.

It may be easily proved that $PM = PN = PP'$ and that a similar property holds for the tangent and normal at P.

Therefore, if two equal rhombs be constructed on PP' having two other sides of each parallel to the axis, their diagonals are tangents and normals at P and P'; and the tangent at one point is parallel to the normal at the other.

Each normal chord divides the other in the ratio 1 : 3.

The chord joining the other ends of the normal chords is parallel to PP' and three times as long.

A line perpendicular to PP' at S, and terminated by this parallel chord and the pole of PP', is divided by S in the ratio 1 : 4.

Hence the locus of the foot of the perpendicular dropped from S on the parallel chord is a right line, whose equation is $x = 9a$.

Hence the envelope of the parallel chord is a confocal parabola, having for its equation $y^2 = 32a(9a - x)$.

It cuts the original parabola orthogonally where it is cut by its evolute.

THE LOGICAL FORM OF GEOMETRICAL THEOREMS.

By Dr. Alexander Macfarlane, Austin, Texas.

In the introduction to the Syllabus of Geometry prepared by the Geometrical Association of England, there is a portion devoted to the logic of the subject; so also in the introduction to Halsted's Elements of Geometry. The Association represents the typical theorem by

If A is B, then C is D;

while Halsted takes

x is y.

I have looked at this subject from the point of view of the *Algebra of Logic*.* The typical form which it suggests, has this great advantage over the preceding, that we can, by means of it, prove the truth of the Rules of Contraposition and Conversion. Let A denote the subject of the theorem (as triangle), x the attribute of the antecedent, y the attribute of the consequent; then the typical theorem is

$$\Sigma A \{x = xy\}, \tag{1}$$

where Σ is used to denote *all*.

The contrapositive is

$$\Sigma A \{1 - y = (1 - y)(1 - x)\}. \tag{2}$$

Now (2) is true when (1) is true, for it is

$$\Sigma A \{1 - y = 1 - y - x + xy\},$$

which, when (1) is true, reduces to an identity.

The converse is $$\Sigma A \{y = xy\}, \tag{3}$$

which does not follow from (1), unless $\Sigma A \{y = x\}$.

The obverse is

$$\Sigma A \{1 - x = (1 - x)(1 - y)\}. \tag{4}$$

When the antecedent of the theorem involves two independent attributes, the type is

$$\Sigma A \{xy = xyz\}. \tag{5}$$

This form has two contrapositives; namely

$$\Sigma A \{x(1 - z) = x(1 - z)(1 - y)\}, \tag{6}$$

and $$\Sigma A \{y(1 - z) = y(1 - z)(1 - x)\}, \tag{7}$$

both of which are true, when (5) is true.

* Principles of the *Algebra of Logic*, by A. Macfarlane, D. Sc., Edinburgh, 1879.

This type has also two converses, and two obverses.

In the works referred to, a rule of conversion is enunciated, but not proved; thus (in Syllabus) "If of the hypotheses of a group of demonstrated theorems it can be said that one must be true, and of the conclusions that no two can be true at the same time, then the converse of every theorem of the group will necessarily be true."

Let the group of theorems be

$$\Sigma A \{x_1 = x_1 y_1\}, \tag{8}$$

$$\Sigma A \{x_2 = x_2 y_2\}, \tag{9}$$

$$\Sigma A \{x_3 = x_3 y_3\}; \tag{10}$$

it is also given that $x_1 + x_2 + x_3 = 1$;

and $y_1 y_2 = 0, \quad y_2 y_3 = 0, \quad y_3 y_1 = 0.$

By adding (8), (9), (10), we get

$$\Sigma A \{x_1 + x_2 + x_3 = x_1 y_1 + x_2 y_2 + x_3 y_3\};$$

$$\therefore \quad 1 = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Multiply by y_1, then $y_1 = y_1 x_2$.

Similarly $y_2 = y_2 x_2$, and $y_3 = y_3 x_3$.

Hence it follows that

$$\Sigma A \{y_1 = y_1 x_1\},$$

$$\Sigma A \{y_2 = y_2 x_2\},$$

$$\Sigma A \{y_3 = y_3 x_3\}.$$

A SHORT DEMONSTRATION OF THE EXPONENTIAL THEOREM.

By Mr. J. M. Schaeberle, Ann Arbor, Mich.

To demonstrate the Exponential Theorem without the aid of the calculus, various methods have been devised by mathematicians; but all of the methods known to the writer are much longer than the one here given, which, for brevity and directness of proof, leaves but little to be desired.

Assuming that the equation

$$a^x = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots \qquad (1)$$

is true for all values of x, then c_0 must equal unity. Squaring (1) we have

$$a^{2x} = 1 + 2c_1x + (2c_2 + c_1^2)x^2 + (2c_3 + 2c_2c_1)x^3 + (2c_4 + 2c_3c_1 + c_2^2)x^4 + \ldots, \qquad (2)$$

while, if in equation (1) we substitute $2x$ in place of x, we have

$$a^{2x} = 1 + 2c_1x + 4c_2x^2 + 8c_3x^3 + 16c_4x^4 + \ldots \qquad (3)$$

Equating the coefficients of the like powers of x in (2) and (3), we at once obtain

$$c_1 = c_1; \quad c_2 = \frac{c_1^2}{2!}; \quad c_3 = \frac{c_1^3}{3!}; \quad c_4 = \frac{c_1^4}{4!}; \text{ etc.};$$

and if the succeeding values of c follow the same law,

$$c_{n-1} = \frac{c_1^{n-1}}{(n-1)!}, \quad c_n = \frac{c_1^n}{n!},$$

which is readily proved by equating the coefficients of the nth power of x in the second members of (2) and (3). Thus

$$2c_n + 2c_{n-1}c_1 + 2c_{n-2}c_2 + 2c_{n-3}c_3 + \ldots + 2c_{\frac{1}{2}(n+1)}c_{\frac{1}{2}(n-1)} \text{ (or } c_{\frac{1}{2}n}^2) = 2^n c_n, \qquad (4)$$

which, if we substitute the assumed values of c_n, c_{n-1}, c_{n-2}, etc., becomes

$$\frac{c_1^n}{n!}\left[2 + 2\binom{n}{1} + 2\binom{n}{2} + 2\binom{n}{3} + \ldots + 2\left(\tfrac{1}{2}\binom{n}{n-1}\right) \text{ or } \binom{n}{\frac{1}{2}n}\right] = 2^n\frac{c_1^n}{n!}. \qquad (5)$$

When n is even, the last term within the [ ] is $\binom{n}{\frac{1}{2}n}$, and when n is odd, the last term is $2\left(\frac{1}{2}\binom{n}{n-1}\right)$. Now, as the value of the expression within the [ ] in equation (5) is always equal to 2^n, it follows that, in general, c_n must always equal $\frac{c_1^n}{n!}$; and hence

$$a^x = 1 + c_1x + \frac{c_1^2x^2}{2!} + \frac{c_1^3x^3}{3!} + \frac{c_1^4x^4}{4!} + \ldots + \frac{c_1^nx^n}{n!} + \ldots$$

A TREATISE ON ALGEBRA.*

This work, as the writers state in their preface, is meant for students who have already an elementary knowledge of algebra and geometry, and as a book of reference for teachers. Among its many merits may be mentioned the completeness of its contents and the exhaustiveness with which each subject is treated. Besides the matter usually treated of in works on algebra, it contains some things not so common, such as infinitesimals and integration, and brings in at the very beginning the idea of function. It also treats of logarithms, continued fractions, etc. in a more extended and helpful manner than usual. The use of checks wherever available and of methods of shortening work, as in detached coefficients, are worthy of note.

The defects of the work as compared with its merits are small. Among them may be mentioned the absence of an index, and the not always clear statement of a proposition or definition. However, the examples accompanying the definitions are very helpful. The work will be of value as a book of reference, but as a text-book it is doubtful whether it can be successfully used, unless preceded by a complete course of elementary collegiate mathematics. F. P. L.

**A Treatise on Algebra.* By Profs. Oliver, Wait, and Jones, of Cornell University. Ithaca, N. Y.: Dudley F. Finch.

SOLUTIONS OF EXERCISES.

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127

A VESSEL of depth a, the top and bottom of which are horizontal planes, is filled with a transparent fluid, the refractive index of which at a depth z below the surface is $1 + z/a$. Two small holes being made in the top, a ray of light enters at one hole, is reflected at the bottom, and emerges at the second hole; show that the distance between the holes must not be greater than

$$2a \log_e (2 + \sqrt{3}). \qquad [\textit{William Hoover.}]$$

SOLUTION.

Let μ be the declivity of the curved ray at depth z. Then the algebraic expression of the hypothesis is

$$\frac{\cos(\mu + d\mu)}{\cos \mu} = \frac{a + z}{a + z + dz},$$

which is reducible to $$\tan \mu \, d\mu = \frac{dz}{a + z}.$$

The first integral of this equation is

$$\log \sec \mu = C + \log (a + z),$$

and as the range is greatest when μ vanishes with z

$$C = -\log a,$$

$$\sec \mu = \frac{a + z}{a}.$$

If in this we put for μ its value $\tan^{-1} dz/dx$ and clear of radicals, we get

$$dx = \frac{dz}{\sqrt{[(a + z)^2 - a^2]}};$$

whence $$x = a \log \frac{(a + z) + \sqrt{[(a + z)^2 + a^2]}}{a},$$

and the greatest range is found by putting $z = a$ to be

$$2x = 2a \log (2 + \sqrt{3}). \qquad [\textit{W. M. Thornton.}]$$

[The proposer sends the same solution; except that the greatest range is determined by the condition that the ray must not suffer total internal reflection on its return to the surface.]

128

A RHUMB line which cuts the meridians at an angle of 30° is projected upon a plane tangent to the south pole. The centre of projection is: 1° at the centre of the sphere; 2° at the north pole; 3° at an infinite distance on the line of the axis. Required the equation of the projection of the curve in each case.

[*Yale Problems.*]

SOLUTION.

Let $\frac{1}{2}\pi - \theta$ be the latitude of any point on the rhumb line, and φ its longitude, counted from the meridian through the intersection of the curve with the equator. Then, if a be the radius of the sphere, ds an element of the rhumb line, and α its constant angle,

$$ds = ad\theta \sec \alpha$$
$$= a \sin \theta \, d\varphi \operatorname{cosec} \alpha.$$

Eliminating ds,
$$d\varphi = \tan \alpha \frac{d\theta}{\sin \theta};$$

$$\therefore \quad \varphi = \tan \alpha \log \tan \tfrac{1}{2}\theta;$$

and the known equation of the rhumb line is

$$\left.\begin{aligned} r &= a \\ \theta &= 2 \tan^{-1} \varepsilon^{\varphi \cot \alpha} \end{aligned}\right\}.$$

Let ρ and φ be the polar co-ordinates in the tangent plane of any projected point; the pole being at the south pole, and φ being equal to the longitude before projection. The equations of the projections are, when $\alpha = 30°$,

1° $$\rho = a \tan \theta = \frac{2a}{\varepsilon^{-\sqrt{3}\cdot\varphi} - \varepsilon^{\sqrt{3}\cdot\varphi}},$$

2° $$\rho = 2a \tan \tfrac{1}{2}\theta = 2a\varepsilon^{\sqrt{3}\cdot\varphi},$$

3° $$\rho = a \sin \theta = \frac{2a}{\varepsilon^{-\sqrt{3}\cdot\varphi} + \varepsilon^{\sqrt{3}\cdot\varphi}}.$$

[*Frank Muller.*]

144

SHOW that the diurnal path of the shadow of the top of a vertical rod on a horizontal plane will be a rectangular hyperbola, if

$$2 \sin \delta = \sin\left(\frac{\pi}{4} + \varphi\right) + \sin\left(\frac{\pi}{4} - \varphi\right),$$

where φ is the latitude of the place and δ the declination of the sun, assumed to be constant. [*R. H. Graves.*]

SOLUTION.

In the spherical triangle formed by the pole P, the zenith Z, and the position of the sun S, we have $SP = 90° - \delta$, $ZP = 90° - \varphi$.

At sunrise and sunset the end of the shadow of the rod will be at infinity, i. e. $SZ = 90°$.

The asymptotes of a rectangular hyperbola are at right angles, and conse-

quently *radii vectores* to the points at infinity which are parallel to them, also enclose a right angle.

The triangles for sunrise and sunset are similar, and the *radii vectores* being parallel to tangents to SZ at Z, Z must equal 135°. Hence by Napier's rules

$$\sin S = \cos 45^\circ \cos \varphi$$
$$= \tfrac{1}{2}[\sin(45^\circ + \varphi) + \sin(45^\circ - \varphi)]. \quad Q.\ E.\ D.$$

[*A. C. Lane.*]

EXERCISES.

151

INTEGRATE the differential

$$\frac{\sin\theta + \cos\theta}{\sin^4\theta + \cos^4\theta}\, d\theta.$$

[*E. Frisby.*]

152

IF r and r' be *radii vectores* in a parabola, and $2f$ the difference of the true anomalies, the area of the parabolic sector between the *radii* is

$$\tfrac{1}{3}\sqrt{(rr')}\sin f.(r + r' + \sqrt{(rr')}\cos f).$$

[*A. Hall.*]

153

THE normals to a parabola at the points P, Q, R pass through the point O and meet the axis at the points P_1, Q_1, R_1; S is the focus and $4a$ is the length of the *latus rectum;* show that

$$PP_1 \cdot QQ_1 \cdot RR_1 = 8a^2 \cdot SO.$$

[*R. H. Graves.*]

154

FIND the locus of the point from which the four normals drawn to an equilateral hyperbola form an harmonic pencil. [*R. H. Graves.*]

155

A TELLS the truth b times out of a; B, whose veracity is unknown, joins A in affirming the occurrence of an event. What is the probability that the event did occur? [*Artemas Martin.*]

156

A HAWK which is 100 yards vertically above a hare that is running with a uniform velocity in a straight line on a level plane, starts in pursuit, and flying continually toward the hare with a uniform velocity, catches the hare at a point 100 yards distant from where the hare was when the hawk started. What is the length of the curve described by the hawk while in pursuit. [*J. E. Hendricks.*]

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